MEASURE THEORY AND INTEGRATION – ALTERNATE FINAL EXAM

	Q1	Q2	Q3	Q4	Q5	Free points	$\sum$
Total score:	15	20	20	20	15	10	100
Score obtained:						10	

1. Assume that $(\Omega, \mathcal{A}, \mu)$ is a probability space. Let (A_n) be a sequence of measurable sets satisfying
$$\sum_{n=1}^{\infty} \mu(A_n) = \infty.$$
Also assume that, for any $n_1 < n_2 < \cdots < n_k$, whenever we have sets $B_{n_1}, \dots, B_{n_k}$ with $B_{n_i} = A_{n_i}$ or $A_{n_i}^c$ for each i, the following holds:
$$\mu\left(\bigcap_{i=1}^{k} B_{n_i}\right) = \prod_{i=1}^{k} \mu(B_{n_i}). \qquad (\star)$$
Show that
$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = 1.$$
Hint. You will need the inequality: $1 - x \le e^{-x}$ for $x > 0$.

2. Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measure spaces. Assume that Ω' is countable and $\mathcal{F}'$ is the σ-algebra of all subsets of Ω'.
   (a) Let $f : \Omega \to \Omega'$ be a function. Show that f is $(\mathcal{F}, \mathcal{F}')$-measurable if and only if
   $$f^{-1}(\{x\}) \in \mathcal{F} \quad \text{for all } x \in \Omega'.$$
   (b) Let f be as in part (a), assume f is $(\mathcal{F}, \mathcal{F}')$-measurable and let $\mathcal{C}$ be the σ-algebra generated by all sets of the form $f^{-1}(\{x\})$, with $x \in \Omega'$. Let $g : \Omega \to \mathbb{R}$ be $(\mathcal{C}, \mathcal{B}(\mathbb{R}))$ measurable. Show that there exists an $(\mathcal{F}', \mathcal{B}(\mathbb{R}))$-measurable function $h : \Omega' \to \mathbb{R}$ such that $g(x) = h(f(x))$ for each $x \in \Omega$.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $f, f_1, f_2, \dots : \Omega \to \overline{\mathbb{R}}$ be measurable functions such that
$$f_n(\omega) \to f(\omega), \quad f_n(\omega) \ge f_{n+1}(\omega) \ge 0, \quad \omega \in \Omega.$$
Show that, if f_1 is integrable, then
$$\lim_{n\to\infty} \int_{\Omega} f_n \, \mathrm{d}\mu = \int_{\Omega} f \, \mathrm{d}\mu.$$
Show that without the assumption that f_1 is integrable, the result need not be true.

4. (a) Let $1 \le p \le q$. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is in $\mathcal{L}^q(\mathbb{R})$ but not in $\mathcal{L}^p(\mathbb{R})$.

(b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is in $\mathcal{L}^p(\mathbb{R})$ for all $p \in [1, \infty)$ but is not in $\mathcal{L}^\infty(\mathbb{R})$.

5. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. Assume that $\mathcal{F}_2$ is the trivial σ-algebra, that is, $\mathcal{F}_2 = \{\Omega_2, \varnothing\}$. Prove that if $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{B})$-measurable, then $f(\omega_1, \omega_2)$ does not depend on ω_2 (that is, $f(\omega_1, \omega_2) = f(\omega_1, \omega_2')$ for all ω_1, ω_2, ω_2'.